Teaching signal and image reconstruction algorithms

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Abstract – **Signal and image reconstruction are among the problems most often faced by whoever works in the broad field of multidimensional signal processing. Thus, an introductory course on this subject would probably be of interest to advanced Electrical Engineering students, specially those interested in telecommunications, information theory, or signal / image processing in general. This paper is a report of my experience in teaching one such course, pointing out some of the difficulties that arose, and discussing the effectiveness of tools such as Matlab or Octave. Some the ideas that are usually explored by the students as part of their assignments are also detailed.**

I. INTRODUCTION

Signal and image reconstruction are among the problems most often faced by anyone working in the broad field of multidimensional signal processing. A relatively large fraction of the signal processing literature is devoted to problems which fall in the scope of multidimensional signal reconstruction, and which includes sampling theory, interpolation, extrapolation, signal and image conditioning, interactive image repair, deconvolution and other inverse problems, the tomographic reconstruction problem, filter design, and much more.

For this reason, an introductory course focusing on multidimensional signal reconstruction is likely to be useful to advanced Electrical Engineering students, specially to those interested in telecommunications, information theory, or signal / image processing in general. In this paper I report on my experience in teaching one course in signal and image reconstruction to final year undergraduates and M.Sc. students at the Electronics and Telecommunications Department of the University of Aveiro.

I discuss some of the difficulties that I have met, and the role played by tools such as Matlab [1] or Octave [2]. I also outline some of the problems that the students are given to solve. These play a fundamental role in the learning process; the lectures devote as much attention to the practical implementation details as to the main theoretical issues, such as convergence and error analysis.

II. OUTLINE OF THE COURSE

The course has been offered to final year undergraduates and M.Sc. students at the Electronics and Telecommunications Department of the University of Aveiro for the past three years. It is assumed that the students have a background in Fourier analysis and digital signal processing in general.

The course aims at preparing the students to deal with situations in which there is incomplete knowledge regarding a desired signal or image, coupled with some *a priori* knowledge (for example, the signal might be bounded in amplitude, or in energy, or perhaps band-limited, or duration-limited). The typical task is to estimate the signal, using whatever knowledge is available. However, it soon becomes clear that there are many other problems which can be solved using techniques similar to those studied during the course.

The course starts with a few mathematical preliminaries that are useful to handle signal and image reconstruction tasks. These preliminary concepts include an overview of some of the most elementary concepts of functional analysis, presented using a simple and suggestive geometrical language, whenever possible. For most of the topics the exposition is restricted to Hilbert spaces. Attention is drawn to the meaning of the concepts in finite-dimensional spaces, and their interpretation in the context of matrix theory.

Following this initial study the concepts of contractive, non-expansive and strictly non-expansive operators, and some well-known fixed-point theorems (the ones due to Banach, Brouwer and Schauder), are introduced. Although the course primary aim is not mathematical rigor, the students are given access to rigorous proofs and reviews of these theorems [3], [4]. The proofs of Brouwer's and Schauder's theorems, which are somewhat lengthy, are not presented in the classes.

The meaning of the theorems in finite-dimensional spaces is examined in the context of matrix theory. The role of matrices with spectral radius less than unity is outlined, and a few well-known iterative methods for solving linear equations are mentioned. This includes the method of Jacobi, the Gauss-Seidel iteration, the JOR and SOR methods, conjugate gradients, and $O(n^2)$ and $O(n \log n)$ specific methods for Toeplitz matrices.

The next step brings the student into contact with reconstruction problems. Instead of probing deeply into a given topic, I try to overview a reasonably large class of reconstruction problems. In this way, hopefully, the course may appeal to as large an audience as possible. The course includes computer simulations and problems, and the students are free to choose from the set of possible tasks those that they find more interesting.

The theory of constrained iterative restoration, based on the approach described in [5], is mentioned. At this stage I discuss applications to deconvolution and the Landau–Miranker theory, which concerns the recovery of companded band-limited signals. The wide-band FM demodulation problem is also addressed, as is the (mathematically similar) nonuniform sampling problem [6], [7].

The band-limited interpolation and extrapolation problems are among the topics to which more time is devoted. The connection between the constrained iterative restoration approach and the Papoulis–Gerchberg [8], [9] algorithm is established as a first step. Modifications of the basic algorithm are addressed as the next step (for example, a nonlinear modification which requires thresholding of the spectrum [10]).

Careful examination of the Papoulis-Gerchberg algorithm leads to Youla's alternating projection method, whose interesting geometrical interpretation [11] is discussed. The method of projections onto convex sets (POCS), which counts so many applications in image processing [12], [13], is a natural follow-up. For all of these topics I strongly emphasize geometrical principles.

A number of additional topics may optionally be discussed (for example, the composite mapping approach [14]). Tomography and algebraic reconstruction methods in tomography are also mentioned, and usually they attract a great deal of attention. The method of Kaczmarz is described in this context, stressing its geometrical interpretation (the method consists of alternating projections onto hyperplanes defined by each of the equations to be solved, and therefore fits very well in the framework of alternating projection algorithms).

Finally, the approach described in [15] is presented, underlining its applications to deconvolution problems, and providing an example of a non-stationary iteration.

The presentation of these topics includes, whenever possible, the discrete finite-dimensional case. This is the case, for example, with the Papoulis–Gerchberg algorithm, which is complemented with some of the results presented in [16]. The disadvantages of the basic iteration are pointed out, and two classes of minimum-dimension methods are proposed to overcome them: the approach described in [17], which leads to algorithms with minimum dimension in the time domain, and the one discussed in [18], which also leads to minimum dimension algorithms, but in the frequency domain. The duality between the two methods [19] is one of the points that is underlined. Some attention is given to the stability of the problems using methods similar to those described in [20].

The problem of detecting the positions of corrupted samples or pixels in band-limited signals or images is also discussed. The modification of the basic Papoulis-Gerchberg iteration suggested in [10] can be used for this purpose, if the time and frequency domain are interchanged. This is a good example of a method originally intended to solve a certain problem, but that can be successfully applied to perform a rather different task. A different algorithm, which reduces the problem to the solution of a set of Toeplitz equations, is also discussed.

To emphasize the practical interest of these techniques I discuss the connection between some of the techniques commonly used in coding theory and signal reconstruction. This connection goes much deeper than could probably be expected: it is shown that band-limited signal reconstruction in the complex field is equivalent to certain error-correcting codes in Galois fields.

III. THE ROLE OF MATLAB OR OCTAVE

Computer simulations and problems are an extremely important part of the course. They allow the student to test the depth of his/her knowledge of the subjects. I try to offer a large choice of computer projects, in order to address specific interests of the students.

It takes time to master some of the concepts, some of which demand a certain amount of mathematical sophistication. Moreover, some of the algorithms are not simple, and implementation using a language like C or Pascal might take some time. Displaying and comparing results is another problem.

To overcome these difficulties I recommend the use of Matlab or Octave, two computer programs endowed with powerful but easy-to-use matrix manipulation capabilities, coupled with a rather complete graphical interface. I do not force the students to use these tools, or in fact any tool in particular. Indeed, I insist that they implement the algorithms using languages, compilers or programs of their own choice. I do recommend that they check Matlab or Octave, since they may allow them to work more rapidly and to obtain graphical representations of the results quickly. Most of the students adopt these tools to implement the algorithms and display and compare the results. The student reports usually include a disk with the code used to implement the algorithms.

Examples of the use of these computer programs as tools to solve reconstruction problems are given in the appendix (these examples are usually given to the students as hints). They address the problem of estimating a subset of the samples of a low-pass signal x with a total of N samples, and whose discrete Fourier transform (DFT) has only $2M+1$ nonzero harmonics. This means that

$$x_k = \frac{1}{N} \sum_{i=-M}^{M} X_i e^{j\frac{2\pi}{N}ik},$$

and so

$$x_k = \sum_{i=0}^{N-1} x_i \frac{\sin(\pi(2M+1)(i-k)/N)}{N\sin(\pi(i-k)/N)}.$$

Two techniques are used to achieve this task. The first is the iterative, finite-dimensional Papoulis-Gerchberg algorithm [16], [21]. The second is the noniterative, minimum dimension method studied in [17]. The code uses a separate command `sincp` to compute the "periodic sinc" function

$$F(x) = \frac{\sin(\pi(2M+1)x/N)}{N\sin(\pi x/N)},$$

that is, the Dirichlet kernel. The Papoulis-Gerchberg method requires a filtering and resampling operation in each iteration, both of which are easily implemented using Matlab or Octave. The minimum dimension method consists in solving the equations

$$u = Su + h,$$

where $u \in \mathbb{C}^n$ is the vector of unknown samples. The elements of the $n \times n$ matrix S are

$$S_{ik} = \frac{\sin(\pi(2M+1)(u_i-u_k)/N)}{N\sin(\pi(u_i-u_k)/N)},$$

whereas the elements of the vector $h \in \mathbb{C}^n$ are

$$h_k = \sum_i x_i \frac{\sin(\pi(2M+1)(i-u_k)/N)}{N\sin(\pi(i-u_k)/N)}.$$

Here, the sum extends to all known samples. The necessary matrix and vector manipulations are easily accomplished using Matlab or Octave.

The appendix also demonstrates the implementation of the nonlinear iterative method that can be used to estimate the number of sinusoids, their frequencies, amplitudes and phases, in a signal of the form

$$f(t) = \sum_{i=1}^{N} a_i \sin(\omega_i t + \theta_i).$$

The same algorithm can be used to detect and correct corrupted samples in oversampled data.

APPENDIX A — OCTAVE CODE

```
function result = Sincp (x, N, M)
%
% Sincp: sin(pi*(2*M+1)*x/N) / (N*sin(pi*x/N))
%
   if (nargin != 3)
      error("Sincp needs 3 arguments\n");
   endif
   [nr, nc] = size(x);
   nels = nr*nc;
   x = reshape(x, nels, 1);
   result = ((2*M+1)/N) * ones(nels, 1);
   i = find(rem(x, N) != 0);
   if (!isempty(i))
      result(i) = sin(pi*(2*M+1)*x(i)/N) ./ ...
         (N*sin(pi*x(i)/N));
   endif
   result = reshape(result,nr,nc);
end

%
% Random low-pass filtered signal x
%

N = 128; % total number of samples
M = 20;  % keep 2M+1 nonzero harmonics

x = rand(1, N);
X = fft(x);
X(M+2:N-M) = zeros(1, N-2*M-1);
x = real(ifft(X));

%
% Sampling set d
%

threshold = 0.5;

d = rand(1, N) > threshold;

%
% Papoulis-Gerchberg iteration
% Result: vector xpg
%

y = x .* d;    % recorded signal
ks = find(d); % known samples

xpg = y;
for i=1:20     % iterations
   %
   % Low-pass filter xpg
   %
   XPG = fft(xpg);
   XPG(M+2:N-M) = zeros(1, N-2*M-1);
   xpg = real(ifft(XPG));
   %
   % Restore the known samples
   %
   xpg(ks) = y(ks);
   %
   % Error
   %
   err(i) = norm(xpg - x);
end

plot(1:N, x, 1:N, xpg);
pause;

plot(err);
pause;

%
% Minimum dimension method
% Result: vector xmd
%

us = find(d == 0);  % unknown samples
n = length(us);

% interpolation matrix S

for i=1:n
   S(i,:) = Sincp(us(i)-us, N, M);
end

% vector h

H = fft(y);
H(M+2:N-M) = zeros(1, N-2*M-1);
h = real(ifft(H));

% find the unknown samples

u = (eye(n)-S) \ h(us)';

% known + unknown samples

xmd = y;
xmd(us) = u';

plot(1:N, x, 1:N, xmd);
pause;

%
% Nonlinear iteration (Papoulis & Chamzas),
% applied to the problem of determining a sparse
% signal "e" with a partially known Fourier
% transform "E" (it is assumed that E(m+2:n-m)
% is known).
%

% k iterations
for i=1:k
   %
   % insert the known samples of E into G
   %
   G(m+2:n-m) = E(m+2:n-m);
   %
   % inverse Fourier transform G to find g
   %
   g = ifft(G);
   %
   % select the threshold L
   %
   L = 0.2 * max(abs(g));
   %
   % positions of the incorrect samples
   %
   u = abs(g) > L;
```

```
  %
  % zero all g(i) that do not exceed L
  %
  g = g .* u;
  %
  % Fourier transform and repeat
  %
  G = fft(g);
end
```

APPENDIX B — COMPUTER PROJECTS

1. Develop a coding strategy for sending critical data over a transmission channel subject to impulsive noise. Use the DFT, and try to detect and correct up to 20 errors per data block.
2. Design a filter subject to the usual frequency response requirements, and to constraints (i) in the amplitude of the time domain impulse response (ii) in the position of the zeros of the time domain impulse response.
3. An oversampled voice signal is sent through the Ethernet, in packets of fixed size. Design an algorithm to recover the signal without error in the event of a packet loss, or excessive delay.
4. Interpolate, extrapolate or predict the evolution of the tides.
5. A slowly-varying signal is aperture modulated. Design an algorithm to recover the original signal.
6. Simulate wide-band FM demodulation.
7. Apply Kaczmarz method to real tomographic data, and display the images obtained. Try convergence acceleration.
8. A band-limited signal is clipped. Design an algorithm to recover the original signal.
9. A band-limited signal is distorted by a nonlinear function and then band-limited. Design an algorithm to recover the original signal.
10. A band-pass signal, with a bandwidth of less than one octave, is quantized using a single bit (the bit preserves the zero crossings of the signal only). Design an algorithm to recover the original signal.
11. Implement deconvolution using the iterative nonstationary technique [15].
12. Implement deconvolution using the iterative constraint / distortion approach [5].
13. Determine a signal of known bandwidth, given a subset of its samples. (use the iterative Papoulis-Gerchberg iteration).
14. Given a subset of the samples of a signal that is the superposition of an unknown number of sinusoids, of unknown frequencies, initial phases, and amplitudes, determine the signal.
15. Determine an image of known bandwidth, given a subset of its pixels (use the 2D iterative Papoulis-Gerchberg iteration).
16. Determine a signal of known bandwidth, given a subset of its samples. (use methods of minimum dimension in the time domain [17]).
17. Determine a signal of known bandwidth, given a subset of its samples. (use methods of minimum dimension in the frequency domain [18]).
18. Determine an image of known bandwidth, given a subset of its pixels (use methods of minimum dimension in the time domain [22], [23]).
19. Determine an image of known bandwidth, given a subset of its pixels (use methods of minimum dimension in the frequency domain [24]).
20. Study the minimum dimension interpolation methods, in the time and frequency domains [19].
21. Study the practical consequences of the ill-posed nature of some band-limited signal and image interpolation problems [20], [23].
22. Design an algorithm to recover an image from the modulus of its Fourier transform and 1 bit of the phase.

IV. CONCLUSION

I have summarized an introductory course on signal and image reconstruction algorithms, which is being offered to final year undergraduates and M.Sc. students at the Electronics and Telecommunication Department of the University of Aveiro.

The subject of signal and image reconstruction was born out of practical problems, its main algorithms are being applied in many fields, helping to solve a large variety of engineering problems. Therefore, computer projects and assignments should play a fundamental role in any course on this subject. These assignments and problems also help in motivating the students, testing their knowledge of the subject, and addressing their specific interests within the broad field of signal and image restoration.

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