Teaching Signal Analysis to Electrical Engineering Students

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Abstract – **The techniques of harmonic analysis play a crucial role in Electrical engineering practice today. Any teacher knows that conveying these techniques is usually far from being an easy task. As for students, they do not exactly regard the subject as the simplest part of their curricula. One of the difficulties faced by the teacher concerns the number of concepts and tools that need to be taught. They include classical Fourier series, the DFT, the Fourier integral, the Laplace and z-transforms, and more. A second difficulty is to convey the role of each of these tools and explain how they are related. A third obstacle appears when simulations or computer experiments need to be carried, since this often occurs at a time when the student does not have sophisticated programming skills. In this paper we report on our experience in teaching this subject, describe the approach that we have adopted, and how to overcome some of these difficulties.**

I. INTRODUCTION

The techniques of harmonic analysis play such a crucial role in Electrical and Electronics engineering theory and practice nowadays that justification of their importance is hardly necessary. As any teacher knows, conveying these techniques is usually far from being an easy task. Students often express views of harmonic analysis that clearly show that they do not regard the subject as the most straightforward part of their curricula.

Anyone that teaches harmonic analysis, including its applications to signal processing and the theory of linear systems, must face several difficulties.

1. The number of concepts and tools that need to be taught is high (browse through [1], [2], for example).
2. The exact role of each of the tools and the way in which they interact is not always easy to describe or summarize.
3. Simulations and computer experiments may have to be performed at a time when the student does not have sophisticated programming skills.

The first difficulty is obvious: the main tools of harmonic analysis required in engineering include the classical Fourier series, the discrete Fourier transform (DFT), the discrete cosine transform (DCT) and other "discrete" transforms, the Fourier integral, the Laplace and z-transforms, the two-dimensional versions of some of these transforms, and more.

Such a diversity of tools leads to the second obstacle, that of conveying the role of each one of them and explaining how they are related.

Often, the teaching of these concepts starts at an early stage in the curricula. This explains the third obstacle: when simulations or computer experiments need to be carried, the student may not have acquired sophisticated programming skills yet.

Overcoming these problems is a hard task. The present work is an attempt to describe the approach that we have adopted, and how it allows some of these difficulties to be removed or at least attenuated.

TABLE I
COURSES RELATED TO THE FIELD OF SIGNALS AND SYSTEMS.

Course	*Year*
Applied Mathematics	2nd
Probability and Stochastic Processes	2nd
Systems Theory	3rd
Signal Processing I	3rd
Control Systems	4th
Signal Processing II	4th

II. THE APPROACH

In this section we summarize the approach adopted to teach harmonic analysis to second and third year students of Electronics Engineering, at the Department of Electronics and Telecommunications, University of Aveiro.

The courses in table I are closely related and part of the broad field of Signals and Systems. In this context, the role of the course Applied Mathematics is fundamental. Its primary aim is to present the basic elements of classical and discrete Fourier analysis, the Laplace and z transforms, and their applications to the study of continuous-time and discrete-time linear systems. Two follow-up courses, Signal Processing I and II, as well as Systems Theory and Control Systems, use and complement these fundamental concepts. A course on Probability and Stochastic Processes is taught in parallel with Applied Mathematics.

Several other courses require a solid background in Fourier analysis. For example, Modulation Theory, Fundamentals of Telecommunications, and, to a lesser extent, Electronics. Again, this background is provided mostly in Applied Mathematics. Not surprising, our chief concern here will be this particular course. However, to keep the discussion down to a reasonable length, we will only outline the main ideas, skipping the details, and concentrating on those aspects that are unusual or perhaps even original. Although we owe a great debt to influential and excellent works such as [1–4], the overall structure of the course follows a pattern that is quite different from that implicit in those works.

The cornerstone of Applied Mathematics is the concept of eigenfunction of a linear, time invariant system. This is the basic concept upon which the presentation is built.

The fundamental concepts (signals, systems) are presented first, of course, and both the continuous-time and the discrete-time frameworks are discussed. Next, the concept of a linear time-invariant system is introduced. Most engineering textbooks define linearity in a purely algebraic way, that is, the system H is called linear if and only if

$$H[\alpha x + \beta y] = \alpha H[x] + \beta H[y], \qquad (1)$$

for any scalars α and β and signals x and y. This definition is not satisfactory because it completely overlooks the topological aspects, and allows for very ill behaved linear systems. To avoid this possibility, aspects such as continuity and closedness are discussed in the course, using very simple and intuitive terms.

To stress the essential role played by these continuity constraints some counterexamples are given. Let x_n be a sequence of signals that are fed to the system H, and let y_n be the sequence of outputs thus obtained. Intuitively, one expects that if x_n converges to, say, x, then y_n converges to $H[x]$. However, there are systems satisfying (1) for which y_n does not always converge, and there are systems for which y_n may converge but not for $H[x]$.

As an example, there is a system satisfying (1) that responds to any continuous signal with the zero signal, and therefore to any sinusoidal signal with the zero signal. Nevertheless, its response to a unit step is the unit step itself! The same is true of any staircase signal.

After this brief digression on the mathematical subtleties of the definition of linearity, the course proceeds to its crucial point. Using elementary arguments and working out directly from the definitions, it is shown that the eigenfunctions of linear time-invariant systems are exponential functions. This is shown for continuous-time systems and discrete-time systems, and leads to the idea of expressing signals as linear combinations of exponential functions.

As a consequence, orthonormal sets of exponentials are built. Links are established with the ideas of linear algebra, and concepts such as eigenvectors and eigenvalues, with which the students are already familiar. At this point it could be discussed, for example, how linear systems defined with respect to finite-dimensional discrete-time signals are mathematically described by matrices. This would introduce the student to the concepts of Toeplitz and circulant matrices.

The task of building orthonormal sets of exponentials leads at once to the classical Fourier series and to the discrete Fourier transform (DFT). Starting from these concepts, we introduce the Fourier integral and the Fourier transform of a discrete signal with an infinite number of samples (and point that the latter is mathematically similar to the classical Fourier series). All of these tools appear not as the result of abstract definitions but as a natural consequence of the steps previously taken, and therefore provide an answer to the often asked question — why Fourier analysis?

The construction of orthonormal sets of exponentials in $[0, N-1]$ (discrete-time case) or in $[0, 1]$ (continuous-time case) is easy. The exponentials e^{at} or e^{an}, with a real, must be excluded because they are everywhere positive, and therefore cannot be orthogonal to each other. This leads to the consideration of exponentials e^{at} or e^{an} with a complex. The simplest possibility is to let a be purely imaginary.

The inner product for complex discrete-time signals in $[0, N-1]$ is

$$\langle x, y\rangle = \sum_{k=0}^{N-1} x_k y_k^*.$$

The signals are orthogonal if the inner product is zero. This immediately shows that two exponential signals

$$x(n) = e^{\mathrm{j}an},$$

$$y(n) = e^{\mathrm{j}bn},$$

are orthogonal if $a-b$ is a nonzero multiple of $2\pi/N$. Since the norm of any exponential signal such as these is $\sqrt{N}$, it is necessary to scale them by $1/\sqrt{N}$ to achieve orthonormality. This yields, of course, the expansion

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k\, e^{\mathrm{j}\frac{2\pi}{N}nk},$$

$$c_k = \langle x, e_k\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)\, e^{-\mathrm{j}\frac{2\pi}{N}nk}.$$

A similar approach is followed to arrive at the classical Fourier series expansion,

$$x(t) = \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{+\infty} c_k\, e^{\mathrm{j}\frac{2\pi}{T}kt},$$

$$c_k = \langle x, e_k\rangle = \frac{1}{\sqrt{T}} \int_0^T x(t)\, e^{-\mathrm{j}\frac{2\pi}{T}kt}\, dt,$$

whose analysis captured the attention of the best mathematicians for over 150 years. The Fourier series of a function of finite energy[1] converges in the mean-square sense[2] to the function. The geometric interpretation of this fact is discussed.

Pointwise convergence and summability are not discussed. However, the subtlety[3] of the pointwise behavior of Fourier series is shown through examples.

The next step is to introduce the discrete Fourier transform, for sampled signals with a finite or infinite number of samples, and the classical Fourier transform. Once again, the analysis is limited to finite-energy signals,[4] and the geometric interpretations are stressed.

[1] Also called a square-integrable function, or a function belonging to L_2.

[2] Mean-square convergence is also called strong convergence, or convergence in norm, or in energy, or just "convergence in the mean".

[3] Engineering books and instructors often forget that a Fourier series need not converge in the pointwise sense to the function that it represents. The series may diverge even if the function is continuous! On the other hand, there are startling functions, such as Weirstrass everywhere continuous but nowhere differentiable function, whose Fourier series converge absolutely and uniformly.

[4] Even in this case there are certain difficulties, due to the limitations of the Riemman integral. These difficulties disappear if the Lebesgue integral is used instead.

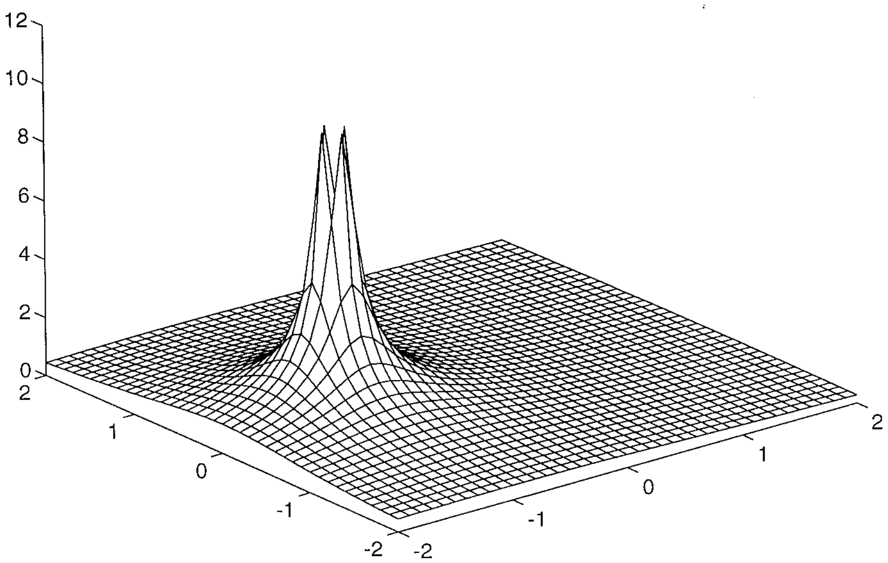

Fig. 1 - Modulus of $F(s) = 1/(1+s)$, the Laplace transform of $f(t) = \exp(-t)$.

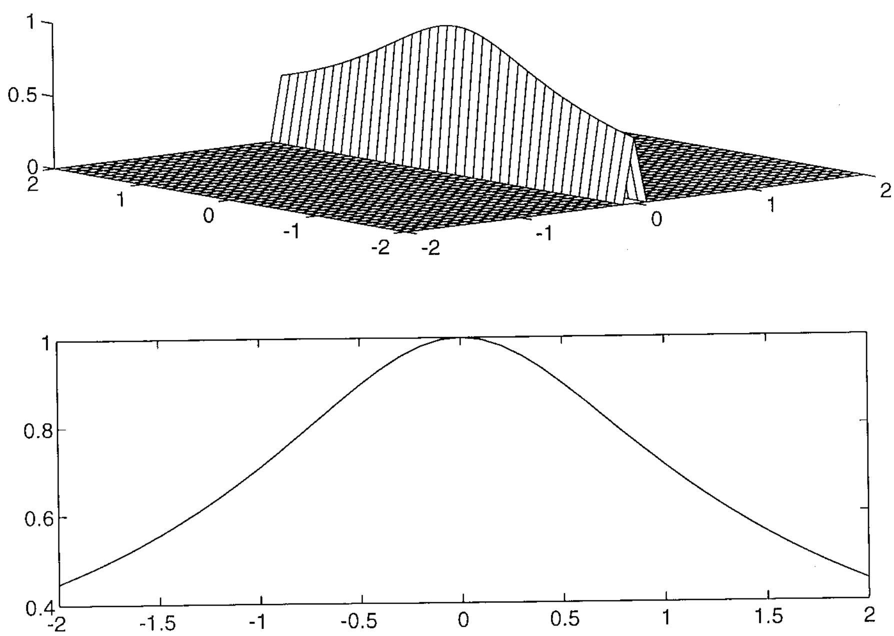

Fig. 2 - A cut of the Laplace transform surface through the imaginary axis $s = j\omega$ yields the modulus of the Fourier transform. Top: the cut on the modulus surface. Bottom: the modulus of the Fourier transform.

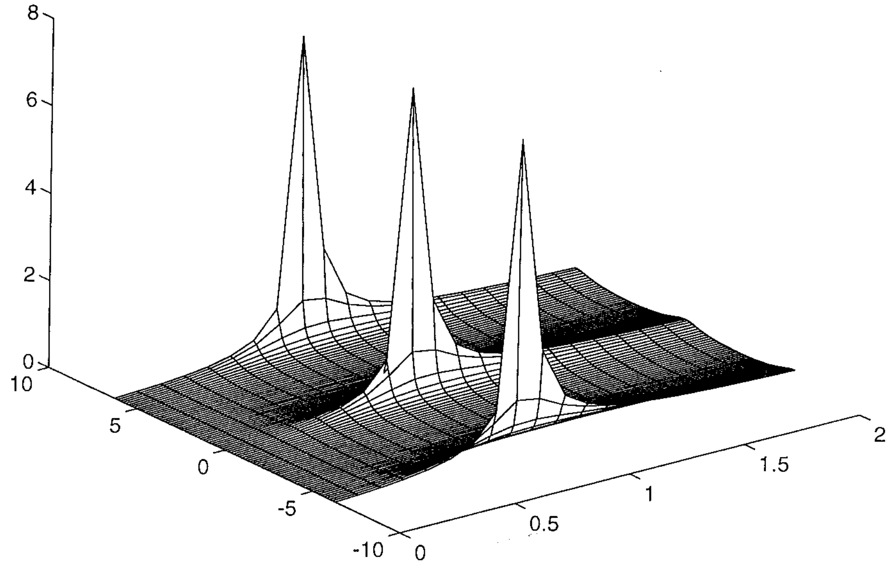

Fig. 3 - Modulus of the Z transform $G(z)$ of $f(k) = \exp(-kT)$, with $T = \pi/16$, plotted as a function of r and ω, $z = re^{j\omega}$.

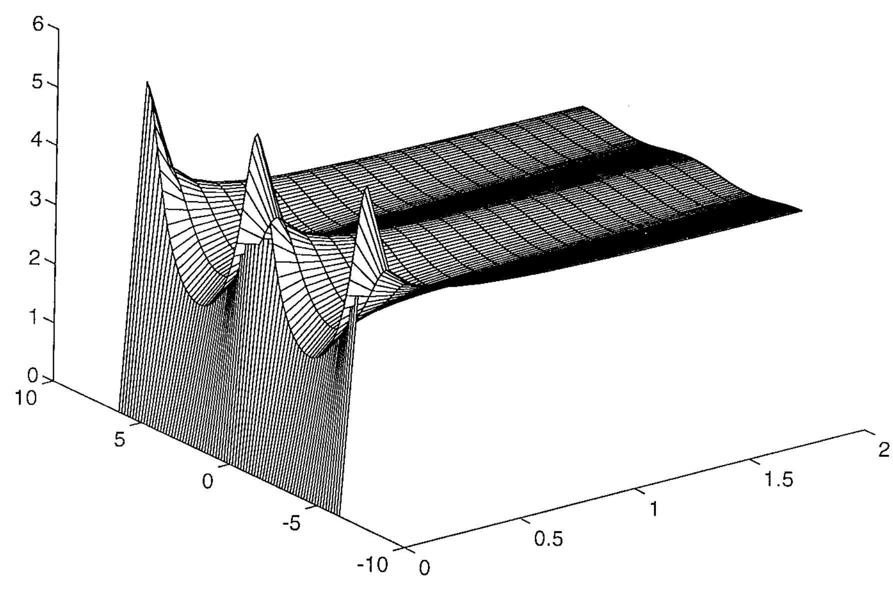

Fig. 4 - Same as figure 3, but now $T = \pi$ leads to serious aliasing.

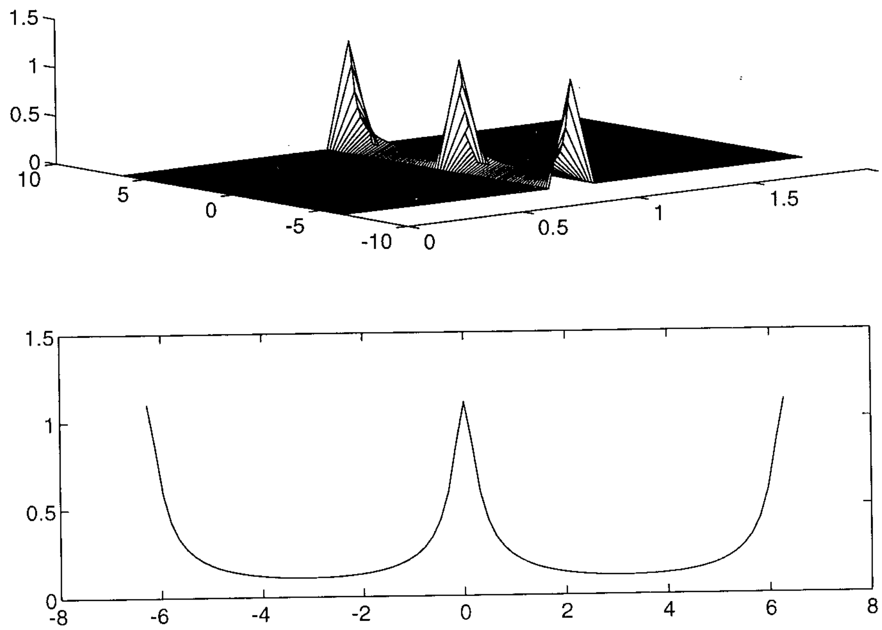

Fig. 5 - A cut of the Z transform modulus surface through the unit circle $z = e^{j\omega}$ yields the modulus of the Fourier transform. In this case, $f(k) = \exp(-kT), T = \pi$. Top: the cut on the modulus surface. Bottom: the modulus of the Fourier transform.

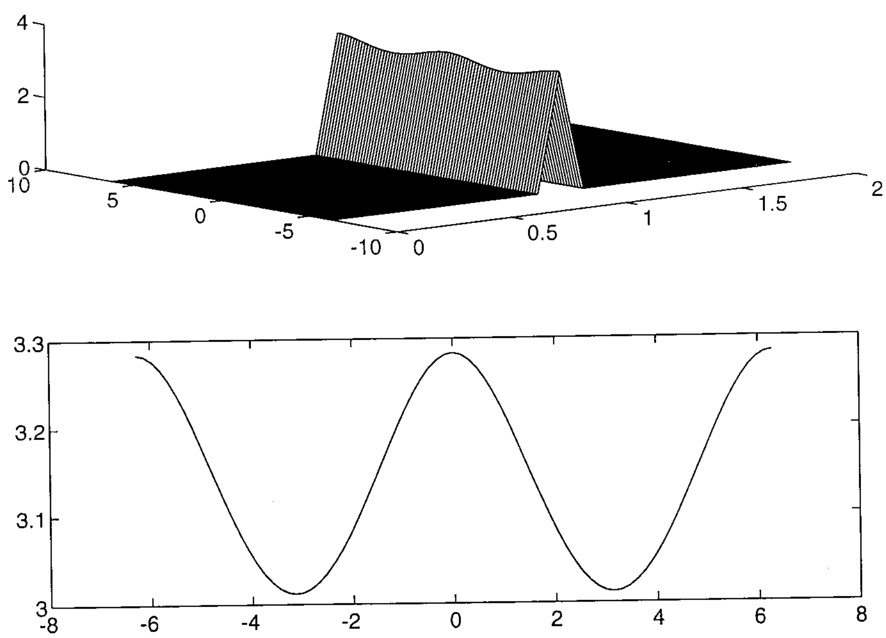

Fig. 6 - Same as figure 5, but now $T = \pi$. Top: the cut on the modulus surface. Bottom: the modulus of the Fourier transform (the scale of the graphics is important to appreciate their significance).

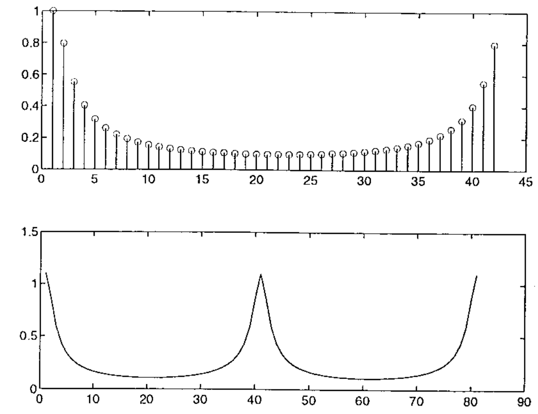

Fig. 7 - Sampling the Z transform on the unit circle yields the DFT. Again, $f(k) = \exp(-kT)$ with sampling period $T = \pi/16$. In this case, the number of samples was 42. Top: the modulus of the DFT samples. Bottom: the modulus of the Fourier transform.

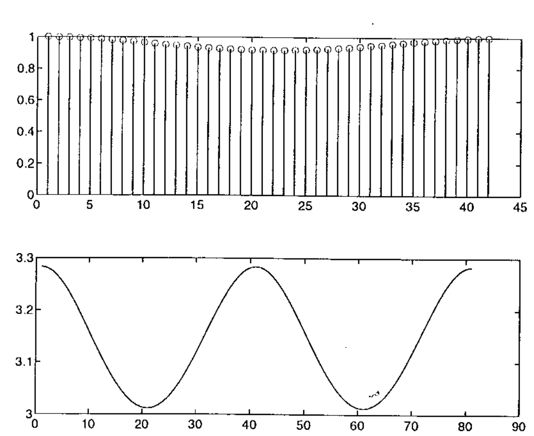

Fig. 8 - Same as figure 7, but now $f(k) = \exp(kT)$, with $T = \pi$.

A. Relations among the transforms

After defining and studying the classical and discrete transforms, the connections among them are examined in detail. For example, the problem of computing approximate values for the coefficients of a Fourier series can be solved using the DFT. Samples of the Fourier transform of a continuous-time signal can also be approximately found using the DFT. This highlights the DFT as a practical tool, and helps in giving a connected and coherent view of Fourier analysis.

B. The Laplace transform and the z-transform

The Laplace transform and the z-transform are studied next. They are introduced as one of the ways to solve the existence problems of the Fourier integral and of the Fourier transform of discrete signals. Their applications to the study of differential and difference equations are examined. Once more, the connections between these two transforms and Fourier analysis are studied. These studies make clear that a linear time-invariant system described by a differential or difference equation is easily described using the Laplace and z-transforms and the concept of "transfer function". The concept of stability is introduced, and the conditions for stability in terms of the transfer function are discussed.

We proceed to recall that linear systems defined on finite-dimensional discrete-time signals are mathematically described by matrices, and investigate the specific form that the matrices take if the system is time-invariant as well. This leads to circulant matrices and the concept of circular convolution. Extension of this investigation to discrete signals with an infinite number of samples leads to the concept of an infinite Toeplitz matrix, and to the convolution sum. Finally, the extension to continuous-time signals lead to convolution kernels and the classical convolution equations. It is shown how the transforms studied bring these equations into much simpler (diagonal) form.

At this point, our main objectives have been fulfilled. The student has a fair background in harmonic analysis and linear system theory and is ready to tackle other problems (the synthesis of linear systems, spectral estimation, multirate systems, and so on). This is done in Signal Processing I and II, as well as in the other courses that have been mentioned. Among the relevant references for these two courses we quote [5–12].

A number of complementary and important topics (the FFT, fast convolution algorithms, equations with a Toeplitz matrix) need also be introduced, of course, but the timing is usually not too critical. They fit in the curricula at any of several points.

III. THE ROLE OF COMPUTER SIMULATIONS

One of the difficulties that we have met when teaching harmonic analysis has to do with the programming skills and general background of the students. It is generally accepted that the omission of practical work (computer simulations and exercises), even at the earliest stage, has serious disadvantages. The students are clearly more motivated when asked to do challenging practical or simulation work, instead of just paper work and routine exercises. However, at the earliest stages in their curricula, they usually have not acquired sophisticated programming skills yet. Consequently, the computer works may easily lead to frustration.

By using tools such as Matlab [13] and Octave [14], the students may concentrate on the problems, and almost forget about the programming. Prototyping and experimenting require comparatively little effort. We hope that the students may come to regard these tools as aids to self-study.

The potential of these tools is not limited to helping the students to carry out the usual simulations (such as filtering, for example). We encourage them to use the tools at their disposal to fully appreciate theoretical concepts as well. For example, the student is asked to use Matlab to visualize the connection between the Laplace and Fourier transforms (figs. 1 and 2), as well as these and other transforms and series (figs. 3,5,7). The role played by the sampling period is also studied (compare with figs. 4,6,8).

IV. CONCLUSION

In this paper we outlined an approach to harmonic analysis aimed at Electrical Engineering students. The difficulties met when teaching concepts and tools such as the classical

Fourier series, the DFT, the Fourier integral, the Laplace and z-transforms, are many. We explained how we try to overcome some of these problems, and the role played by computer simulations. Our experience has been positive and encouraging, and confirms the feeling that practical experimentation and simulation at the earliest stages provide an invaluable contribution towards a deeper understanding of harmonic analysis.

APPENDIX — MATLAB CODE

```
% modulus of the Laplace transform of
% f(t)=exp(-t), that is,  H(s)=|1/(s+1)|

[xs,ys]=meshgrid(-2:0.1:2);
H=abs(1 ./ (xs+j*ys+1));
figure(1)
mesh(xs,ys,H)

% cut a section through the imaginary axis
% to display the Fourier transform modulus

L=size(H,1);
H1=zeros(L);
L=ceil(L/2)
H1(:,L)=H(:,L);
figure(2)
subplot(211),mesh(xs,ys,H1);
subplot(212),plot(ys(:,L),H(:,L))

% modulus of the z transform z/(z-a), with
% a = exp(-T)

T=input("sampling period:");

r=0:.1:2;
N=40
t=-2*pi:2*pi/N:2*pi;
[rs,ts]=meshgrid(r,t); % (r,theta) grid

z=rs.*exp(j*ts);
H=abs((T*z)./(z-exp(-T)));
figure(3)
mesh(rs,ts,H)

% cut a section through the unit circle
% to display the Fourier transform modulus

figure(4)
[i,j]=size(H)
H1=zeros(i,j);
H1(:,11)=H(:,11);
subplot(211),mesh(rs,ts,H1);
subplot(212),plot(ts(:,11),H(:,11))

% DFT of the sampled signal (42 samples)

n=0:1:41;
x=exp(-T*n)
xf=abs(fft(x));
a=max(xf);
figure(5)
subplot(211),stem(xf./a)
subplot(212),plot(H(:,11))
```

REFERENCES

[1] A. V. Oppenheim and R. W. Schafer. *Discrete-Time Signal Processing*. Prentice-Hall, Englewood Cliffs, New Jersey, 1989.

[2] L. B. Jackson. *Signals, Systems and Transforms*. Addison-Wesley, Reading, Massachusetts, 1991.

[3] A. V. Oppenheim and R. W. Schafer. *Digital Signal Processing*. Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

[4] L. R. Rabiner and B. Gold. *Theory and Applications of Digital Signal Processing*. Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

[5] A. Antoniou. *Digital Filters: Analysis and Design*. McGraw-Hill International Editions, New York, 1979.

[6] R. N. Bracewell. *The Fourier Transform and Its Applications*. McGraw-Hill International Editions, New York, 1986.

[7] D. Childers and A. Durling. *Digital Filtering and Signal Processing*. West Publishing Company, Saint Paul, 1975.

[8] A. V. Oppenheim, A. S. Willsky, and I. A. Young. *Signals and Systems*. Prentice-Hall, Englewood Cliffs, New Jersey, 1983.

[9] L. B. Jackson. *Digital Filters and Signal Processing*. Kluwer Academic Publishers, Dordrecht, 1986.

[10] A. Papoulis. *Signal Analysis*. McGraw-Hill International Editions, New York, 1987.

[11] S. Haykin. *Adaptive Filter Theory*. Prentice-Hall, Englewood Cliffs, New Jersey, second edition, 1991.

[12] B. Widrow and S. D. Stearns. *Adaptive Signal Processing*. Prentice-Hall, Englewood Cliffs, New Jersey, 1985.

[13] The Mathworks, Inc. *The Student Edition of Matlab: Version 4: User's Guide*. The Matlab Curriculum Series. Prentice-Hall, Englewood Cliffs, New Jersey, 1995.

[14] J. W. Eaton. Octave: a high-level interactive language for numerical computations. URL: ftp://ftp.che.utexas.edu/pub/octave, 1995.