Superoscillations

Paulo J. S. G. Ferreira[1], Dae Gwan Lee[2]

1 — Department of Electronics, Telecommunications and Informatics & IEETA, University of Aveiro
2 — Department of Mathematical Sciences KAIST, South Korea

Superoscillations are oscillations of a signal at a rate faster than its maximum frequency. They are a key component of Aharonov's weak measurement formalism in quantum theory with important applications in imaging, superresolution and antenna theory. A paper published in Nature (Rogers et al. 2012) argues that superoscillation-based imaging has unbeatable advantages over other technologies".

We have given a new construction of superoscillations. By taking a set of signal values as the adjustable parameters, we showed how the signal can be made to superoscillate by prescribing its values on an arbitrarily ne and possibly nonuniform grid. Surprisingly, the superoscillations can be made to occur at a large distance from the nonzero samples of the signal. We gave necessary and sufficient conditions for the problem to have a solution. Depending on the number of signal parameters, the problem can be exactly determined, underdetermined or overdetermined. We described the solutions in each of these situations and succeeded in describing the connection with oversampling and certain variational formulations. We also showed that it is possible to construct superoscillations by constraining the signal and its derivative. This allows much better control of the shape of the superoscillations. We found that for any given bandwidth, no matter how small, there exists a unique signal of minimum energy that satisfies any combination of amplitude and derivative constraints, on a sampling grid as fine as desired. We found the energy of the solution, for regular or irregular grids. The flexibility gained by having two different types of constraints makes it possible to design superoscillations based only on amplitudes, based only on derivatives, or based on both. In the last case, the amplitude and derivative sampling grids can be interleaved or aligned. We explored this flexibility to build superoscillations that cost less energy.

The paper considers a numerically difficult task: the synthesis of superoscillations. Minimum energy solutions are attractive for applications because (i) the minimum energy solution is unique (ii) it has the smallest energy cost (iii) it may yield a signal with amplitude as small as possible. On the negative side, minimum energy solutions depend heavily on cancellation and involve expressions with very large coefficients. Furthermore, these coefficients have to be found by solving equations that are very ill-conditioned. Surprisingly, shows that by dropping the minimum energy requirement practicality can be gained rather than lost. The method produces superoscillating signals with coefficients and condition numbers that are smaller by several orders of magnitude than minimum energy solutions, yet yields energies close to the minimum. The method has another important property: it yields the superoscillatory signal that maximises the energy concentration in a given set, which may or may not include the superoscillatory segment. The paper introduces translation in time by as a design parameter. We succeed in giving an explicit closed formula for the condition number of the matrix of the problem, as a function of the translation number (see the figure). This enabled us to find the best possible condition number, which is several orders of magnitude better than otherwise achievable.

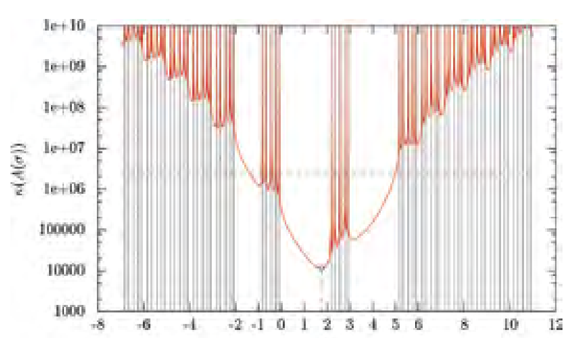

FIGURE 1
The condition number of the problem as a function of the translation . The optimum condition number is marked. Our methods yield the corresponding, enabling the design of optimally stable superoscillations.