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Learning and teaching symmetry by creating ceramic panels with Escher type tessellations

Aprender e ensinar simetria através da criação de painéis cerâmicos com pavimentações do tipo das de Escher

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Abstract:

Elliot Eisner (1933-2014), a pioneer in arts education, suggested that an artistic approach to education could improve its quality and lead to a new vision for teaching and learning. This is true for any subject, including mathematics. Some topics of the mathematics school curriculum make a perfect setting for a deeper contribution of art to education and allow for a complete symbiosis between the teaching of mathematics and an artistic education. One such topic is the study of symmetry and isometry, present throughout the school mathematics curriculum from elementary to secondary levels. We believe that the learning and teaching of symmetry and isometry can be greatly facilitated by taking the role of an artist and creating works of art, eventually inspired by renowned artists. In this paper, we present some results of a professional development course for mathematics teachers where the participants created ceramic panels using the same techniques as the Dutch artist M.C. Escher did in his tessellation drawings.

Keywords: professional development; art and mathematics; symmetry; mathematical education.

Resumo:

Elliot Eisner (1933-2014), pioneiro em educação artística, sugeriu que uma abordagem artística para a educação poderia melhorar a sua qualidade e levar a uma nova visão do ensino e a aprendizagem. Esta ideia aplica-se a qualquer área de ensino, incluindo a matemática. Alguns tópicos do programa de matemática adequam-se perfeitamente a interligação mais profunda da arte com a educação e permitem uma simbiose completa entre o ensino da matemática e uma educação artística. Um desses tópicos é o estudo da simetria e da isometria, presentes ao longo do programa de matemática, desde o ensino básico até ao secundário. Acreditamos que o processo de aprendizagem e de ensino da simetria e da isometria pode ser fortemente facilitado ao se assumir o papel de artista e criar obras de arte, eventualmente inspiradas por artistas bem conhecidos. Neste artigo, apresentamos alguns resultados de uma

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ação de formação contínua para professores de Matemática onde os participantes criaram painéis cerâmicos replicando as mesmas técnicas que o artista holandês M.C. Escher usou nos seus desenhos de pavimentações.

Palavras-chave: formação contínua; arte e matemática; simetria; educação matemática.

Resumen:

Elliot Eisner (1933-2014), pionero en educación artística, sugirió que un enfoque artístico para la educación podría mejorar su calidad y llevar a una nueva visión para la enseñanza y el aprendizaje. Esta idea se aplica a cualquier área de enseñanza, incluyendo las matemáticas. Algunos tópicos del programa de matemática se adecuan perfectamente a la interconexión más profunda del arte con la educación y permiten una simbiosis completa entre la enseñanza de las matemáticas y una educación artística. Uno de estos tópicos es el estudio de la simetría y de la isometria, presentes a lo largo del programa de matemáticas, desde la enseñanza básica hasta el secundario. Creemos que el proceso de aprendizaje y de enseñanza de la simetría y la isometría puede ser fuertemente facilitado al asumir el papel de artista y crear obras de arte, eventualmente inspiradas por artistas bien conocidos. En este artículo, presentamos algunos resultados de una acción de formación continua para profesores de Matemática donde los participantes crearon paneles cerámicos usando las mismas técnicas que el artista holandés M.C. Escher hizo en sus diseños de pavimentaciones.

Palabras clave: formación continua; arte y matemáticas; simetría; educación matemática.

Introduction

Today's society is one of high complexity and continuous-change. On the one hand, it is plural and fragmented and, on the other, global and interconnected. The rapid growth of information and communication technologies imposes equally rapid growing challenges to our society. These can only be satisfied with a transdisciplinary education, capable of forming planetary, solidary and ethical citizens, able to face the challenges of the present times (Morin, 2002). Morin advocates the incorporation of daily problems into the school curricula and the interconnection of knowledge, as opposed to fragmented teaching.

Mathematics is an area considered to be fundamental in today's world. Its importance in day-to-day life and hence in everyone's education is irrefutable (Earls & Holbrook, 2007) and it can even be said that this is the discipline that is both at the base and at the top of the scientific culture chain (Lima, 2004). Considering the importance that mathematics plays in today's society, it has been the subject of concern for many authors and researchers due to the enormous academic and educational failure in several countries, including Portugal (Araújo & Cabrita, 2012, Niss, 2003). One of the problems identified in the failure of mathematics education is the lack of motivation that students feel towards the subject. Motivation is an essential factor in any learning since the quality of learning is not only related

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to the capacity to learn, but also to the level of motivation we have to carry out that same learning (2006, p.41).

In this context, the proposal of new transversal and interdisciplinary methodologies appears as a promising solution to the identified problem. An artistic approach to education has been suggested by several authors as a means to improve its quality and achieve a new paradigm for teaching and learning. A pioneer in this field was Elliot Eisner (1933-2014) who inspired Dietiker (2015) to apply his ideas to mathematical education. In order to implement changes in education methodologies, it is necessary to first train teachers in this new paradigm, so that through them one reaches the students, in schools. In this sense, the present work involved, in the first place, Mathematics teachers of primary and secondary education.

Art and mathematical education

Mathematics classes in schools are usually described by students as being uninteresting and boring. This scenario is not a recent one: more than 10 years ago, Gadanidis & Hoogland (2003, p.489) described the mathematics that is experienced in school as being *flat-lined* and, over 30 years ago, Davis & Hersh (1981, p.169) described mathematics classes as being *dry as dust*. Dietiker (2015, p.1) refers that, in spite of several curricular reforms (mostly conformed to content revisions), students commonly experience mathematics as being *uninspiring and dull*.

Many researchers have tried to provide solutions and suggestions to deal with this problem. Elliot Eisner, a pioneer in arts education, believed that a conception of practice rooted in the arts should contribute to the improvement of both the means and the ends of education (Eisner, 2002, p.4).

More striking than it was, in 2002, the public concern about educational productivity of schools has led to 'tight controls, accountability in terms of high-stakes testing and pre-specification of intended outcomes' (Eisner, 2002, p.6). The natural consequence, as Eisner puts it, is "to tighten up, to mandate, to measure, and to manage" (p.6). This seems to be worsening the problem instead of solving it. Eisner suggested that new values in education are needed and believed that educational challenges can be met by taking an artful approach: "the aim of education ought to be conceived as the preparation of artists" (p.8). By artists, he does not mean just the professional ones, but also "all individuals who have developed the ideas, the sensibilities, the skills, and the imagination to create work that is well proportioned, skilfully executed, and imaginative, regardless of the domain in which an individual works" (p.8). His new vision for teaching and learning is one in which "more importance is placed on exploration than on discovery, more value is assigned to surprise than to control, more attention is devoted to what is distinctive than to what is standard" (p.16). He defends an educational culture that

> *has a greater focus on becoming than on being, places more value on the imaginative that on the factual, assigns greater priority to valuing than to measuring, and regards the quality of the journey as more educationally significant than the speed at which the destination is reached. (p.16)*

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Engaging students in the classroom is a fundamental task for an effective and fulfilling learning experience. In the arts, engagement tends to be secured by the aesthetic satisfactions obtained from the work itself (Eisner, 2002, p.14). The work being created presents natural challenges which are related to part of these satisfactions:

> *Materials resist the maker; they have to be crafted and this requires an intense focus on the modulation of forms as they emerge in a material being processes. This focus is so intense that all sense of time is lost. The work and the worker become one. (p.14)*

Aesthetic satisfaction is indeed a fundamental aspect of human life. Beauty, as described by Scruton (2009, p.12), "is a real and universal value, one anchored in our rational nature, and the sense of beauty has an indispensable part to play in shaping the human world". As Plato and Plotino already considered, beauty is a fundamental value which we look for in itself without any further justification; beauty compares to truth and kindness, ultimate values which justify our rational choices (Scruton 2009, p.16). However, beauty does not apply solely to great works of art. Just like Eisner considers a broader concept of what defines an artist, Scruton believes that beauty also has a broader range of application: "The minimal beauties are far more important to our daily lives, and far more intricately involved in our own rational decision, than the great works which (if we are lucky) occupy our leisure hours" (p.24).

Eisner (2002) believes that the feeling of vitality and the surge of emotion we feel when experiencing an art can be embedded in the ideas teachers explore with students and in the appetite for learning that teachers stimulate: "In the long run, these are the satisfactions that matter most because they are the only ones that ensure that what we teach students will want to pursue voluntarily after the artificial incentives so ubiquitous in our schools are long forgotten" (p.14-15). Just like Eisner considers that "the arts are, in the end, a special form of experience" (p.14), Dewey (1934) considers that learning is also a special form of experience, writing "all genuine education comes about through experience" (p.25). However, he also noted that not all experiences are educative: educational experiences are those "that live fruitfully and creatively in subsequent experiences" (p.28).

Although many students do not have a positive attitude towards mathematics, most teachers have a natural like for mathematics. Labelle (2000), professor of mathematics, says, "I like mathematics because it is beautiful, full of surprises, and gives me complete freedom of thought" (p.10). Beauty brings pleasure and has to do with people's feelings (Scruton, 2009). Feelings of surprise and beauty are emotional; doing mathematics is emotional (Gadanidis & Hoogland, 2003, p.488). Furthermore, emotions are central to our lives and the way we experience them. It is a challenge for teachers to find ways to continue experiencing their natural pleasure in doing and teaching mathematics while simultaneously deal with a student's unpleasant reactions towards mathematics.

Today's world is changing faster than ever before. New challenges emerge every day. Students must be prepared to face the uncertainty and constant change of the future. However, the density and rigidness of most mathematics curricula head in the opposite direction of what is needed. Taleb (2014) points out the importance of the unexpected in the present days. He argues that humans tend to focus on what they already know, preferring the particular and

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disregarding the general. Like in many other contexts, school curricula reflect our tendency to control and predict. Eisner (2002) argues that, although understandable at a practical level, this tendency must be inverted: "Opening oneself to the uncertain is not a pervasive quality of our current educational environment. I believe it needs to be among the values we cherish. … How can the pursuit of surprise be promoted in a classroom?" (p.11).

Professional development for mathematics teachers

Today's fast changing pace allied to the complexity of mathematics teaching practices place extraordinary demands on practicing teachers. Professional development is an important means to help teachers find ways to meet these demands. For almost thirty years, the mathematics education community has made a continuous effort to change teaching practices towards being more cognitively demanding, conceptually oriented and student centred (Heyd-Metzuyanim, Tabach & Nachlieli, 2016).

School subjects like mathematics pose additional challenges because many students have a negative attitude towards the subject and the general school results are not satisfactory in many countries. In Portugal, for instance, the mathematics school results call for special attention. The most recent Programme for International Student Assessment results (PISA, 2015) placed Portugal in the 29nd position among the 72 intervening countries. These results, although better than previous ones, call out for attention from all decision makers, teacher trainers and teachers themselves.

Countries with low performance results in school mathematics try to improve their results through a variety of strategies. As noted by Pournara et al. (2015), teachers' knowledge matters in all learning contexts, in particular in countries with low social-economic status. Improving teachers' knowledge and teaching skills is therefore a priority. For instance, in New Zealand, a Mathematics and Science Taskforce was created in order to provide advice on how to improve the teaching of mathematics and science in New Zealand schools. Higgins & Parsons (2009) reported that this taskforce highlighted a number of high priorities including "…improve the professional skills, knowledge, and confidence of teachers; provide resources and professional development for teachers to assist and support them in implementing the curriculum" (p.232). Similarly, in 2006 the Portuguese government implemented a national wide plan for the teaching of mathematics ("Plano de ação para a Matemática" – Plan of action for mathematics), for all grades of basic and secondary education. This plan consisted of six main actions including a massive professional development of teachers from all over the country (with special incidence in basic education), and the implementation of projects to promote the success in Mathematics ("Plano da Matemática I" 2006-2009 and "Plano da Matemática II" 2009-2011) (DGE, 2018; Oliveira & Fernandes, 2011). Unfortunately, due to the financial and economic crises that emerged in Portugal in 2010, financial restrictions forced the government to draw back on the investment made. Nevertheless, professional development courses for teachers are mandatory for career progression and, in general, teachers participate regularly in these courses and are aware of their importance. Indeed, as Borko et al. (2010) wrote, "if we want schools to offer more powerful learning opportunities for students, we must offer more powerful learning opportunities for

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teachers – opportunities that are grounded in a conception of learning to teach as a lifelong endeavour and designed around a continuum of teacher learning" (p.548).

A lot of research has been done in the past years on teacher's professional development. Borko et al. (2010) highlight the recent shifts in professional development methodologies. These shifts are somewhat related to recent shifts in the prominence of ideas about the nature of cognition, learning and teaching. As Borko refers, professional development should move away from a rigidly structured in-service training model towards approaches grounded in classroom practice and involving the formation of professional learning communities (p.548). Indeed, as Higgings & Parsons (2009) wrote, "professional learning opportunities for teachers need to be situated in the teacher's context of practice and relevant to the teaching and learning needs of teachers and students" (p.232). These learning opportunities should focus on the real context of the classroom and be integrated into the teachers' everyday work.

Symmetry and Escher type tessellations

"Symmetry has a wide appeal; it is of as much interest to mathematicians as it is to artists, and is as relevant to physics as it is to architecture", wrote David Wade in the introduction of his book about symmetry (Wade, 2006, p.1). Symmetry is somehow a universal principal that can be found everywhere, from the microscopic world of matter to the mysterious behaviour of galaxies, not forgetting all living creatures and most human conceptions. In spite of being completely entangled with the presence of asymmetry (just like order goes hand in hand with disorder), the fact is that humankind has a need to look for regularity and patterns in order to better understand the world.

The study of symmetry and isometry is now an important part of the mathematics curriculum at the elementary and secondary school levels. It is very appealing to use applications and examples of symmetry/isometry in the real world to address the topic in the classroom. The visual arts are a nourishing field for that purpose. Symmetry/isometry can be found in paintings, ceramic pieces, pavements, textile works, iron works, sculptures, architecture, etc.

In a broad sense symmetry has to do with some form of repetition. Geometrically speaking, symmetry has to do with congruence or periodicity and is strongly related to the concept of isometry. Since there are four types of isometries in the plane, there are also four types of symmetry in the plane – reflection, rotation, translation and glide reflection. A symmetry of a shape is an isometry that maps it onto itself (Teixeira 2015, p.52). A shape is said to be symmetrical (or present symmetry) if it has at least one symmetry besides the trivial one (the identity).

The set of all symmetries of a figure together with the composition operation form a group. A group of symmetry can be classified according to the types of symmetries in the group. If there are no infinitely small translations or rotations in the group, it is said to be discrete. Discrete symmetry groups in the plane are classified into 3 large categories: rosette, frieze (periodic pattern along one direction) and wallpaper (periodic pattern along two directions) groups (Martin, 1982). The definition of each of these categories can be made in several equivalent ways, one of which relates to the number of fixed points of the symmetries involved.

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The most basic isometry, the identity, fixes all the points of the plane. A reflection fixes the reflection axis, point by point. A non-trivial rotation fixes only one point, the centre of rotation. Finally, a non-trivial translation or a glide-reflection have no fixed points.

Rosette groups are discrete groups of symmetry in the plane which have at least one fixed point. These groups have a finite number of symmetries. Rosettes only have rotational and/or reflection symmetry.

Frieze and wallpaper groups are discrete groups of symmetry in the plane which have no fixed points. These groups have an infinite number of symmetries. Friezes exhibit only 7 types of symmetry groups and wallpaper patterns exhibit 17 symmetry groups. For the purpose of this paper, we shall only consider wallpaper groups. There are several designation systems for wallpaper groups and we shall use one from crystallography. The 17 types are labelled: p1, p2, pm, pg, cm, pmm, pmg, pgg, cmm, p4, p4m, p4g, p3, p3m1, p31m, p6 and p6m. The numbers relate to the highest order of the rotational symmetries. Letters m and g indicate mirror (reflection) and glide reflection symmetry. Letters p and c have to do with the type of cell (primitive or centred). A detailed description of all of the groups can be found in many geometry text books, as for instance Martin (1982). Classifying a wallpaper pattern is not a straightforward task, at first sight. However, Washburn and Crowe (1988) provide a simple flowchart to classify any wallpaper pattern which makes the task accessible to anyone who is familiar with the basics of symmetry. For the purpose of this paper, we are only interested in using the classification in order to illustrate the variety of symmetry groups obtained in the examples given in the next section.

Wallpaper patterns and tessellations are closely related. Fathauer (2008) defines a tessellation as a collection of shapes that fit together without gaps or overlaps to cover the infinite mathematical plane (p.19). When a tessellation forms periodic patterns (in two directions) it becomes an example of a wallpaper pattern. Fathauer says that, in a general sense, a tessellation covers any surface, not necessarily flat or infinite. In this sense, tessellations are found in all cultures from very early on. In this paper, we shall only consider periodic tessellations on the (infinite) plane.

A real world tessellation is a tiling made of physical materials such as ceramic tiles or fabrics. Many real world tiling have a decorative purpose and many other have a practical use, such as providing durable and water-resistant pavements. Some combine both such as the traditional Portuguese pavements of sidewalks and squares (see Teixeira (2015) for the analysis of symmetry in Azorean sidewalks for instance) and most pavements used in Ancient Rome and in Islamic art (such as in the decorative tiling of the Alhambra palace, in Spain).

A prominent contribution to the exploration of artistic tessellations was given by M.C. Escher, a Dutch graphic artist who lived from 1898 to 1972. Escher visited the Alhambra Palace in Granada, Spain, in 1922 and became interested in periodic tessellations. Despite not being a mathematician, his work is strongly influenced by mathematical concepts and Escher made his own mathematical incursions in order to achieve his artistic purposes. He produced the first of his 137 wallpaper designs gathered in his notebooks in 1926, but it was only in 1937, after his second visit to Alhambra in 1936, that he started to master his own techniques to produce such tessellations. In 1958, he published a commissioned essay on his tessellation work, The

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Regular Division of the Plane. He remained fascinated by tessellations until the end of his life: his last numbered design was produced in 1971 (Fathauer, 2000, p.5). Escher's notebooks are extensively reproduced and analysed by Schattschneider (2004) and many of his works can be seen at the official website of M.C. Escher (http://www.mcescher.com/gallery/). None of Escher's engravings consists exclusively of a wallpaper pattern. These patterns were kept only in his notebooks as drawings. However, he makes extensive use of his tessellations in is metamorphosis and cyclic works (Ernst, 2007, p.24).

One interesting and challenging aspect of Escher's periodic tessellations is the use of recognizable (figurative) motives for the basic tiles (birds, fish, horses, boats, beasts, reptiles,...). He even regretted that the Alhambra drawings had no such figurative motives. In his texts, he said that "it is a pity that the Islamic religion forbids the representation of images; Alhambra mosaics are limited to the use of abstract geometric figures" (Ernst, 2007, p.41).

One other important aspect of Escher's tessellations is colouring. In a tessellation identical tiles may or may not have the same colour. Colours may affect the symmetry properties of the pattern. One needs to specify whether the colours are part of the tiling or just part of its decoration. Visually, a colouring that preserves symmetry, highlights it. A perfect colouring is one where any symmetry results in a permutation of the original colours (Fathauer, 2008). The 4 colour theorem ensures that 4 colours are enough to guarantee that no adjacent tiles share the same colour. Escher makes a wise use of colouring, using mostly perfect colourings with contrasting colours that highlight the beauty of his works.

The technique used by Escher to create his periodic tessellations basically consisted of the following: start with a tessellation of simple polygons (as for instance a square or triangle grid); transform the edges of the polygons of the grid using certain adequate models based on plane isometries; fill in each tile with decorative motives and colour. Many different transformation models can be applied and each one will result in a distinct wallpaper pattern with different symmetrical properties (belonging to different symmetry groups). Robert Fathauer (2008) wrote an extensive applied book on this type of tessellations intended for use in the classroom. He provides numerous templates and examples using different models, obtaining a wide variety of possible tessellations. He also gives many hints on practical issues related to this type of procedure. The simplest transformation is one based on translations. For instance, starting with a square, one can produce the following transformation based on two translations:

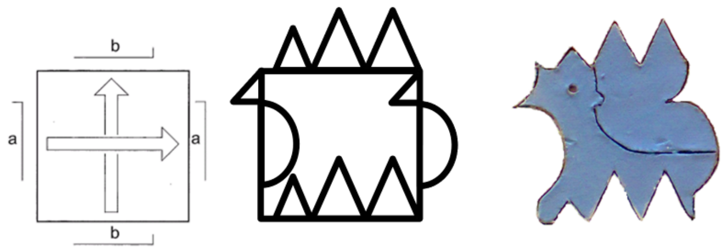

Fig. 1 Example of a square transformation using translations

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The resulting tessellation can be seen in figure 2.

This type of technique, although not common in schools, appears in some school textbooks, in activity books and also in computer software. In particular, two activity books are worth mentioning for their rich mathematical content and artistic value: Alex Bello, a British writer, who has greatly contributed to the popularization of mathematics together with the mathematical artist Edmund Harriss, published a colouring book which contains three examples of periodic tiling resembling Escher's works (Bellos & Harriss, 2015); Ana Weltman, an American teacher with a passion for teaching children about maths, published a maths activity book based on many different mathematical contents, some of which beyond the scope of the maths curriculum but very interesting (like fractals and visual illusions). Weltman's book also contains examples of Escher type tessellations, to be built by the reader (Weltman, 2015). Also worth mentioning is the software Tessemaniac, created by Kevin Lee (http://www.tesselmaniac.com/tess/Home.html), which allows the user to create his own tessellations through a variety of models. This program is available free as a demo version, however with a limited range of models and facilities.

Method

Methodological options

The research questions underlying this study are:

1. To what extent can the teaching and learning of isometries and symmetries be improved by an arts education approach?
2. To what extent can an applied study stemming from the analysis of art works by renowned artists help to improve the teaching and learning process of isometries and symmetries?

As a consequence, the following objectives were defined:

To analyse the influence of an artistic approach and the use of art works from renowned artists in the development of geometric competences on symmetries and isometries in mathematics teachers, towards:

- a more solid appropriation and application of the geometric concepts involved;
- the development of teaching methodologies that can elicit more positive attitudes towards mathematics in general and geometry in particular, in students.

In this perspective, a mixed case study was developed (quantitative and qualitative, based on a logic of complementarity), grounded on a pragmatic paradigm and case study design (Carmo & Ferreira, 1998; Ponte, 2006; Yin, 1994). The study was undertaken in a higher education institution of central Portugal involving the participants of a professional development course

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intended for mathematics teachers. The teacher responsible for the course in question is simultaneously one of the researchers of this study.

In order to develop this experience, the techniques of inquiry, direct observation and documental collection were applied and the following instruments were used: final questionnaire; field notes and interviews.

Description of the course

The professional development course addressed in this paper is part of a set of professional development courses for mathematics teachers promoted over the last several years following the ideas presented in the previous sections of this paper. In these courses we try to integrate the activities into the teachers' everyday work, while at the same time provide an opportunity to both learn and have a fulfilling and gratifying experience. Following the ideas presented by Borko et al. (2010), these courses are intended to be "opportunities grounded in a conception of learning to teach as a lifelong endeavour", that should be both pleasurable and rewarding. At the same time, we try to move towards Eisner's conception of a practice rooted in the arts. Teachers take on the role of an artist: they are given time to explore, to create and to surprise themselves.

The professional development course in focus was titled *Let's do ceramic tiles with mathematics* and took place in a Portuguese university, in the central region of the country, in 2013, from January 20th to April 4th. Like all professional development courses for Portuguese teachers, it was acknowledged by the national scientific and pedagogical council for teacher's professional development (Conselho Científico-Pedagógico da Formação Contínua), and was registered with the number CCPFC/ACC - 67053/11. This course consisted of 25 hours of contact between all participants and had the collaboration of a ceramist who taught part of the course. The course had 14 participating teachers who taught mathematics from grades 1 to 12. Given the wide range of grades taught and the specificity of each level of teaching, different activities were proposed for teachers of different levels (elementary and secondary).

In the first part of the course, some concepts were provided or recalled and related activities were carried out. The participants performed several tasks intended to deepen their knowledge on isometry/symmetry and to prepare them for the applied project they had to develop during the second part of the course. For instance, after recalling the four types of isometries in the plane, the participants were asked to explore the composition of reflections and after some experiments they were lead to conclude that any isometry in the plane can be obtained from the composition of at most three reflections.

Still during the first part of the course the participants explored the 17 wallpaper groups of symmetry and used Washburn and Crowe's flowchart to classify given examples. In addition the participants were asked to analyse in several ways a set of tessellations by M. C. Escher and by R. Fathauer. Colouring issues, figurative aspects, methods of constructing the tessellation and symmetry properties (including the identification of the symmetry group) were all considered

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in this analysis. Some particular tasks were also performed in order to develop the necessary skills to produce their own tessellations following Escher's technique. Finally, a large set of models used to produce tessellations was explored.

In the second part of the course the participants were asked to develop an individual project that put the theoretical concepts into practice. Each participant was challenged to create their own piece(s) of art using a wide variety of models (randomly assigned), and if possible create figurative type designs, as Escher did. The final result was a glazed and coloured ceramic panel by each participant.

The participants produced their tessellations starting with polygons that tile the plane. As described in the previous section, the basic procedure consists of starting from these polygons, performing deformations on some of its sides and transferring them to the other sides using isometries. The following representation of the plane isometries were used:

Reflection  Rotation 

Glide reflection  Translation 

The polygons used were squares, rectangles, rhombuses, parallelograms, kites, triangles and hexagons. Many different transformations can be performed to produce tessellations. Altogether 17 different models were explored. Some of the participants used the Tesselmaniac software to produce their designs.

The resulting tessellations were then used to produce ceramic panels with the help of Maria da Purificação Barros, a ceramist who taught this part of the course. At the same time, the other trainer (and author of this paper) joined the group and created her own ceramic panel. In the end, the participants had to analyse the symmetries of their work pieces and classify the wallpaper patterns using Washburn and Crowe's flowchart.

Results

Participants' outcomes

In this section, photos of a selection of the ceramic panels produced are shown. For each work, the underlying basic polygon and the transformations applied to produce the proto-tile (basic tile that results in the whole tessellation) are presented. Also provided is the wallpaper symmetry group of the tessellation (not considering colouring) which was part of the assignment and gives an idea of the variety of symmetry types obtained. The author of each work is identified in each case.

The first two examples given in figure 2 follow the same transformation procedure using only translations applied to two sides. They are based on a square and a parallelogram. The transformation applied to obtain the first example is given in figure 1.

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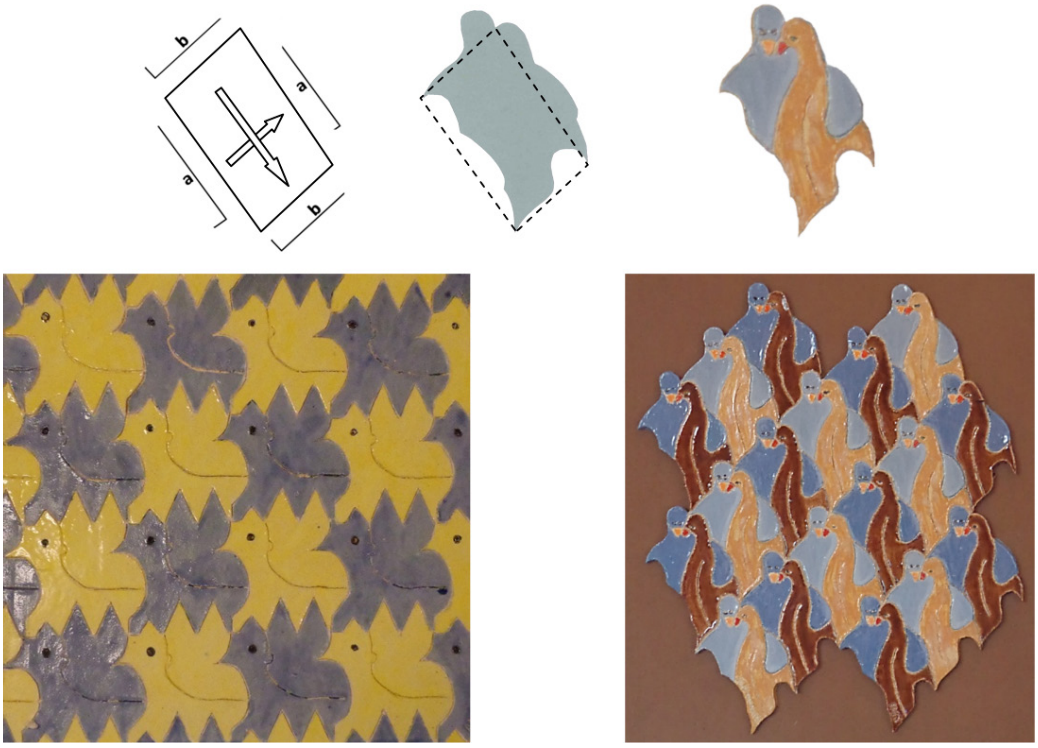

Fig. 2 Panels by Ana Cristina Martins (left) and Andreia Hall (right). Symmetry group (SG): p1

The example in figure 3 is based on rotations applied to the sides of a rhombus.

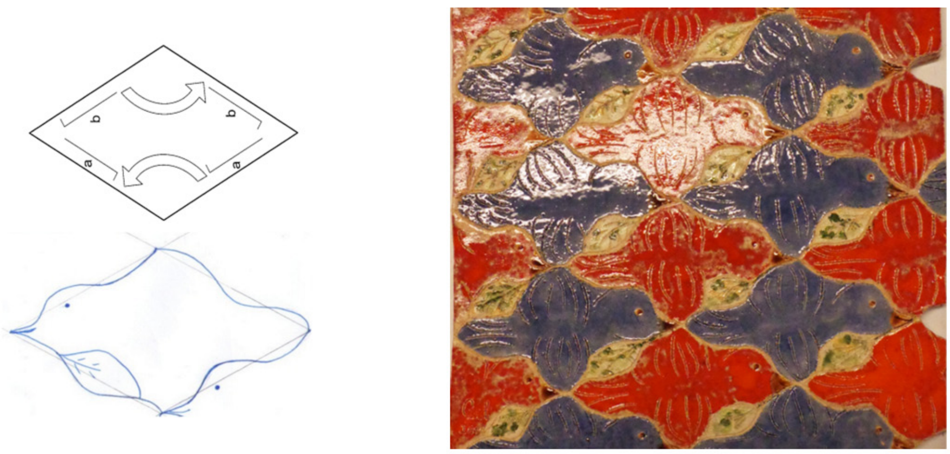

Fig. 3 Panel by Cecília Tavares. SG: p1

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The example in figure 4 is based on translations and rotations of halved sides over a rectangle. The example in figure 5 is based on rotations and reflections over a rhombus (two adjacent equilateral triangles). The examples in figure 6 are based on glide reflections over a rectangle.

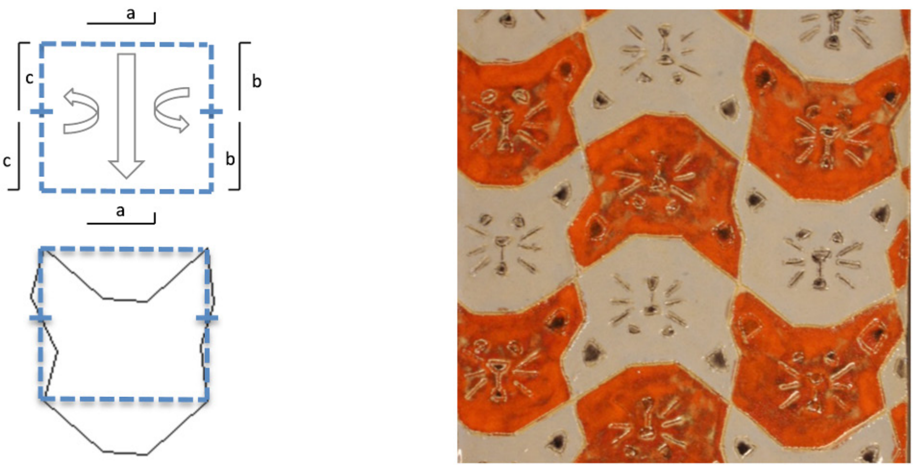

Fig. 4 Panel by Carla Merendeiro. SG: p2

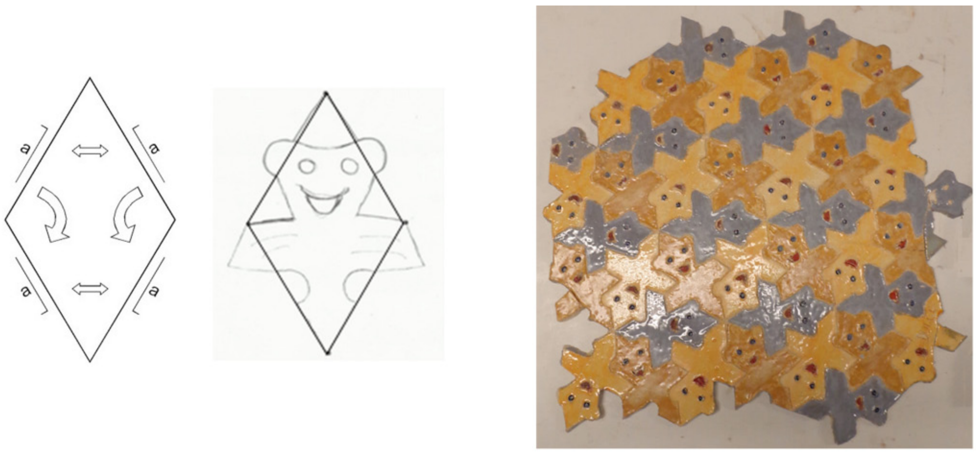

Fig. 5 Panel by Dulce Mesquita. SG: p31m

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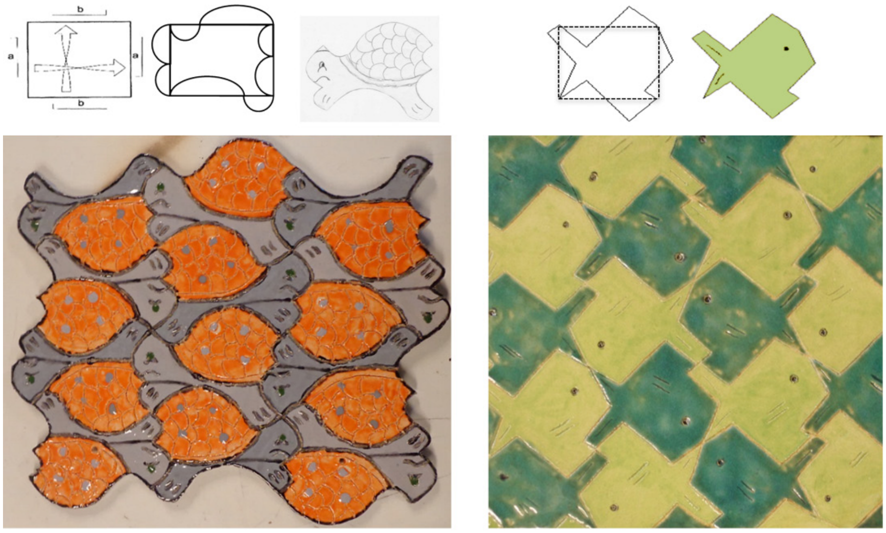

Fig. 6 Panels by Ana Cristina Martins (left) and Carla Merendeiro (right). SG: pgg

The examples in figure 7 are based on reflections and glide reflections over a square. Note that the resulting symmetry group depends on the way the tiles are placed next to each other. Thus, the example on the right has a different group although the proto-tile was designed using the same type of transformation.

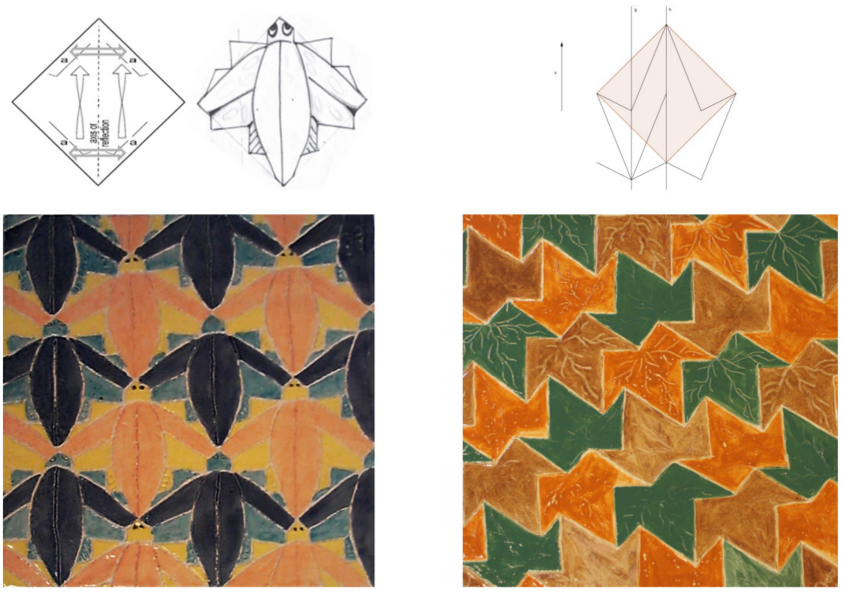

Fig. 7 Panels by Dora Cristina Alfaiate (left; SG: cm) and Maria Ester Morais (right; SG: p2)

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The example in figure 8 is based on rotations over a right triangle.

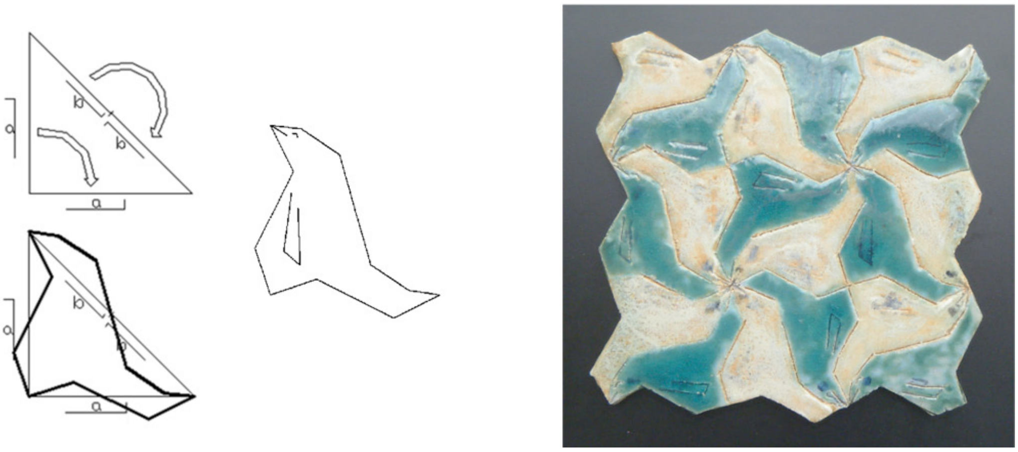

Fig. 8 Panel by Luísa Pinheiro. SG: p4

The example in figure 9 is based on rotations over an equilateral triangle.

Fig. 9 Panel by Dulce Mesquita. SG: p6

The example in figure 10 is based on rotation and glide reflection over an equilateral triangle. For this tessellation, two symmetric proto-tiles are needed.

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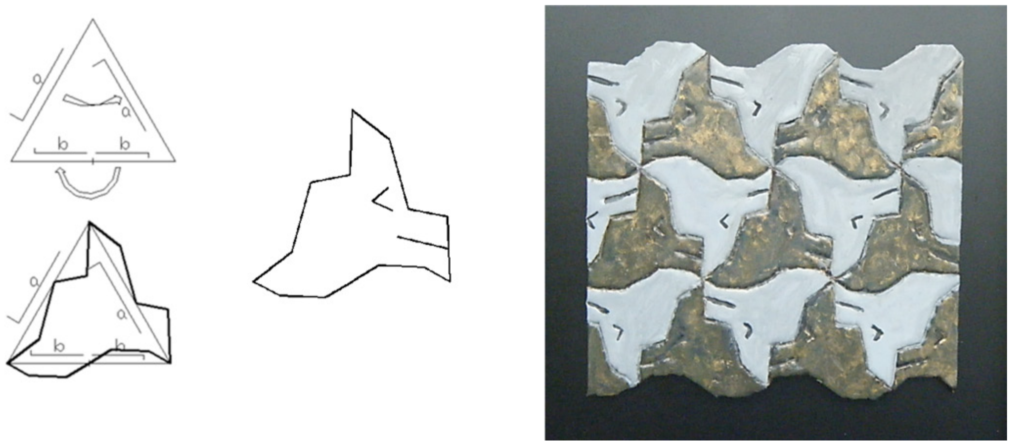

Fig. 10 Panel by Maria Manuela Pinheiro. SG: pgg

The example in figure 11 is based on reflections and rotations over half sides.

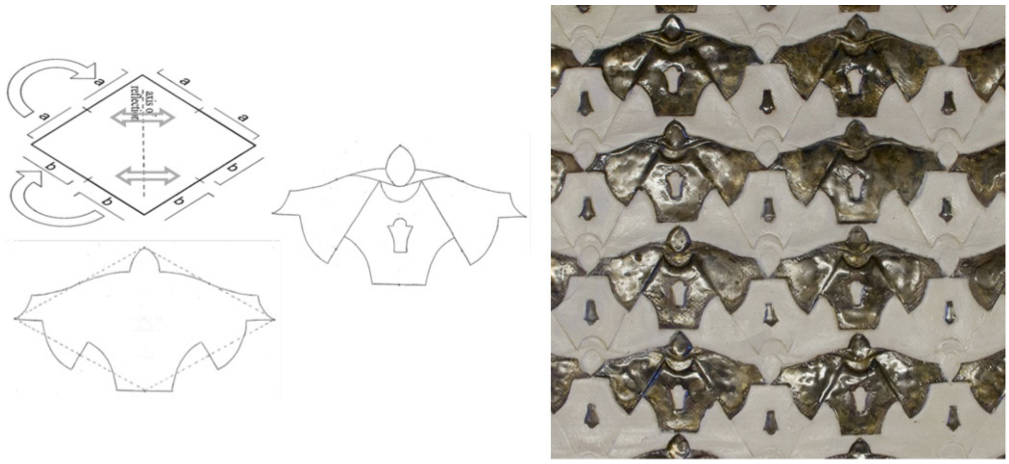

Fig. 11 Panel by Filipe Monteiro. SG: pmg

The last example given in figure 12 is based on reflections over a square and for this tessellation two different proto-tiles are needed.

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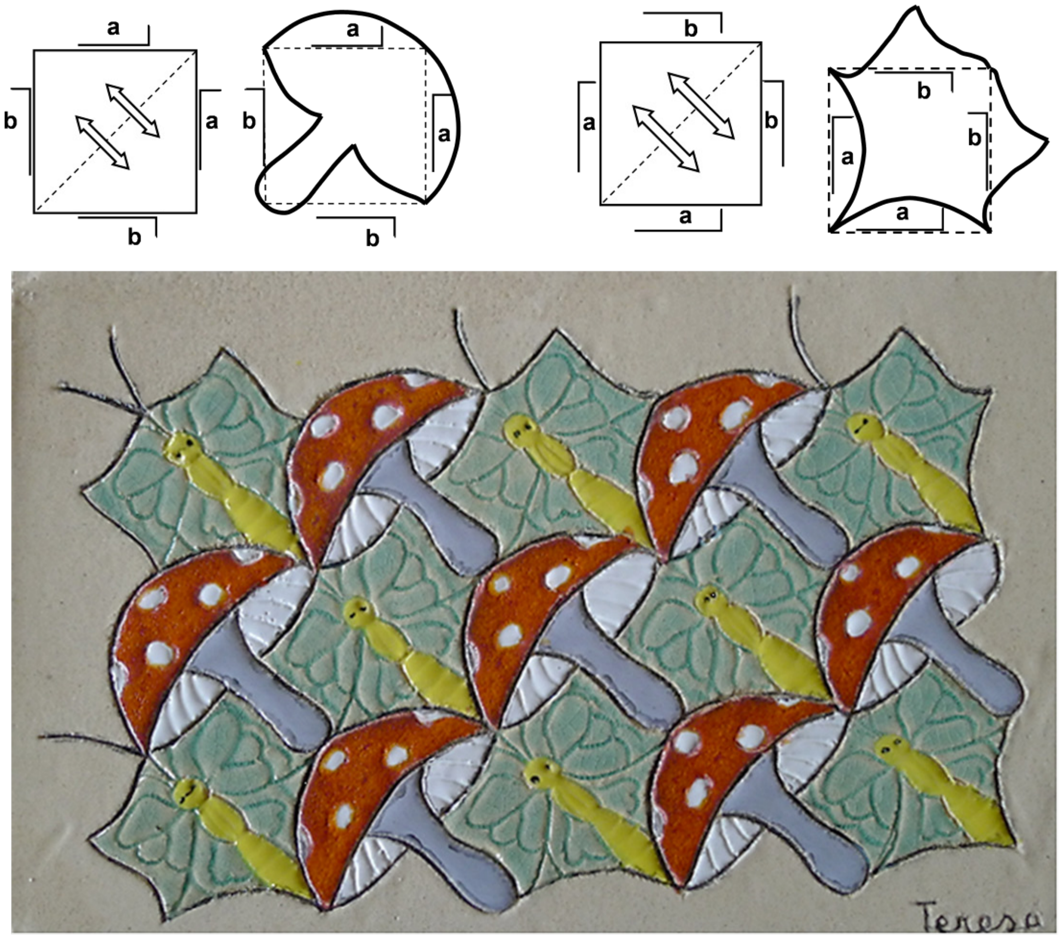

Fig. 12 Panel by Teresa Mena. SG: pm

Other results

This paper intends to analyse the influence of an artistic approach in the development of teaching methodologies that can elicit more positive attitudes in students towards mathematics in general and geometry in particular. To reach this goal, direct observation and field notes were the main instruments used. Through direct observation it can be concluded that the participants were deeply interested in performing the tasks, especially those based on real world examples such as Escher's works. From the field notes, it was observed that more than once, the participants expressed their amazement as they realized how complex and yet marvellous Escher's tessellations are. In order to analyse the influence of an artistic approach on the development of geometric competences towards a more solid appropriation and application of the concepts involved, direct observation was used. It was observed that the

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geometric concepts involved were gradually mastered, as desired, while at the same time the participants were exploring and having contact with works of art. Whenever possible, the tasks were given in a set up appropriate for use in the classroom. The participants were given a wide variety of tasks that could be used with their students, interlacing mathematics with art.

The fact that they had a concrete goal to achieve – the development of an individual work of art – was an important source of motivation for and engagement with all the tasks. As the tasks were performed, we observed an increasing willingness to perform their own projects, though they also realized how difficult the challenge was. The difficulty felt was never a motive of demotivation but rather recognition of the artistic qualities of M.C. Escher which cannot be achieved by everyone, much less in such a short time.

As previously mentioned, this course consisted of 25 contact hours between participants and supervisors and it was not expected that the participants would dedicate relevant extra time to the course. However, we could observe the engagement of the participants with the course amongst other things through the work they developed at home. It was not a matter of completing unfinished work, but rather wanting to explore other alternatives and achieve further results in their challenges. During the development of their individual projects, the participants had to create two different designs (using different models) but only one of these designs was expected to be used to create a ceramic panel. However they were told that, if they had time,, they could use both designs and produce two different panels. As a result, some participants created more than two designs from which to choose for the ceramic panels and all of them produced two ceramic panels. For this they all had to spend extra time beyond the 25 hours initially planned, but this was done exclusively of their own will and they were really excited about their projects. Definitely, in this part of the course, many participants surprised us with their high creativity and commitment in the projects.

A questionnaire was answered by the participants at the end of the course. Teachers were asked to answer through a five point Likert scale (1-"not at all"; 5-"very much") to the following questions:

1. Were the topics of the course pertinent?
2. Did the course update your knowledge on the subject?
3. Did the course allow you to develop skills in a curricular and extracurricular context?
4. Did the course meet your expectations?
5. Were the activities interesting?

Through the analysis of the questionnaires: Questions 1, 2 and 4 were answered as "very much"(5) by all the respondents except one who answered "much"(4); Question 3 was answered as "very much"(5) by all but two respondents who answered "much"(4); Question 5 was answered as "very much" (5) by all respondents. Therefore, it can be concluded that there was a positive influence of the artistic approach used in the course towards a more solid appropriation and application of the geometric concepts involved and in the development of teaching

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methodologies that can elicit more positive attitudes towards mathematics. The obtained results are consistent with the filed notes registered by the teacher during the professional development course. Not only the participants' curiosity, motivation and enthusiasm was great and enduring but also their performance regarding both the preparatory geometry tasks and the artistic projects revealed a good appropriation of the concepts involved.

Participants were also asked to make suggestions regarding what needs to be changed or kept in future editions of the course. All except two respondents said that nothing should be changed in the course. The two who suggested changes wished there were more sessions for the course. We highlight one of the comments which said that: "In future courses, it is important to keep the collaborative spirit, the availability of the instructors and the enthusiasm and joy present all the way through".

Finally, they were asked to express what they liked and disliked most in the course.

Regarding what they liked most, several answers were given such as having liked the way the course was given and the quality of the materials provided. Transcribed below are some of the answers closely related to the goals of this research.

What I liked most in this course was…

- to use mathematics in new contexts, in this case ceramics. For me it is very important to **learn to use mathematics in real-life contexts**, since I work with students with alternative curriculum having other mathematic classes such as "Mathematics for Life" or "Mathematics and Reality".

- the exploration of Escher's drawings, the creation of new tessellations from the given models, and its transposition to ceramic pieces. The **interlocking of mathematics with art** was also interesting.

- having had an **interlaced theoretical/practical learning experience**.

- to **transfer knowledge** of isometries to the preparation of the tiles.

- having **created** figures using symmetry and working with clay.

- to be able to **update and deepen my knowledge**; .... I reinforce the above, stating that the "activity with the letter F" [where the participants explored the composition of reflections to obtain rotations, translations and glide reflections] was considered very interesting and served as a starting point, for exploring and deepening the programmatic contents in the classroom.

In this professional development course, an artistic approach to teaching and learning was facilitated, since each participant was given the opportunity to create a concrete piece of art that naturally produces an aesthetic satisfaction which in turn leads to a deep engagement with the work itself. As referred before, Eisner (2002) considers that this strong engagement is so intense that the sense of time can be lost. As we directly observed and registered in our

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field notes, during this course the participants had the pleasure to witness, more than once, the loss of their sense of time while engaging with their creations.

The topics covered in the course were symmetry and isometry. In the real world, symmetry can be found in many forms of artistic expression: paintings, ceramic pieces, pavements, textile works, iron works, sculptures, architecture, etc. In addition to taking profit from an artistic approach to the subject itself, it was observed that teachers felt increasingly confident with their arts and crafts skills. This is certainly an added value which will be useful to them in the classroom.

The results described above show that an arts education approach stemming from the analysis of renowned artists' works and focused on the development of creative multidisciplinary projects led to a positive improvement in the teaching and learning process of isometries and symmetries.

Final remarks

The professional development course described in this paper sought to establish a strong interconnection between mathematics and the visual arts, both by the analysis of M.C. Escher's works and by involving the participants in multidisciplinary projects to create ceramic works involving mathematics, aesthetics and artistic creation.

The participants showed great commitment in the accomplishment of the proposed tasks and projects. Their performance (throughout the entire course) showed that they deepened and acquired new knowledge associated to the topics of isometry and symmetry. The phase of development of artistic projects was particularly rewarding, since the participants were involved in every step of the process, dedicating with pleasure all the necessary time to its accomplishment. Most of all, the participants showed enthusiasm and joy during the course and these are main ingredients for successful teaching and learning.

Overall, the activities developed have proved to be successful examples of interdisciplinary methodologies that bring into the teaching of mathematics usual procedures in the teaching of the arts, associated to the process of artistic creation.

We believe that these teachers have been strengthened in their capacity to develop multidisciplinary tasks and projects with their students. The methodologies used, involving the interconnection between different areas, promote a positive attitude towards mathematics and thus foster the motivation to learn it.

The results of the present research show that the selected approach contributes to improve the teaching and learning process of isometries and symmetries present in the curriculum of elementary and secondary education. The questionnaire filled out at the end of the course showed that the overall evaluation of the course was very positive and allows us to conclude that its goals were accomplished.

Overall, it can be concluded that both objectives established for this research were positively met since the artistic approach followed during the course (including the analyses of renowned

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artists' works and the development of artistic projects) helped the teachers develop their geometric competences concerning isometry and symmetry in a more solid appropriation and application of the geometric concepts involved, and further contributed to the development of teaching methodologies that can elicit more positive attitudes in students towards mathematics in general and geometry in particular.

Regarding the first research question - To what extent can the teaching and learning of isometries and symmetries be improved by an arts education approach? – we can say that this study provided a positive answer to it. In regards to our second research question – To what extent can an applied study stemming from the analysis of art works by renowned artists help to improve the teaching and learning process of isometries and symmetries? – this study allows us to conclude that the use of applied examples (in this case art works from renowned artists) for the exploration of mathematical concepts improves the teaching and learning process by increasing the motivation and also the cultural background of teachers and students.

Many researchers have considered similar research-questions in different contexts and advocate that mathematical and general education can benefit from an arts education approach (Dietiker, 2015; Burton et al., 1999; Rooney, 2004; Short, 2001). Today's society values education and believes Science and Technology are crucial areas for improving the quality of life. Engaging students within STEM fields (Science, Technology, Engineering and Mathematics) is considered a priority by most governments. Curiously, the need to incorporate Arts in these priority fields has also been acknowledged and the acronym STEM has been converted into STEAM by many (Ghanbari, 2015; Graham & Brouillette, 2016; Pomery, 2012; Sousa & Pilecki, 2013).

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